ULB-TH/05-23, hep-th/0512105

Three dimensional origin of Gödel spacetimes and black holes

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We construct Gödel-type black hole and particle solutions to Einstein-Maxwell theory in 2+1 dimensions with a negative cosmological constant and a Chern-Simons term. On-shell, the electromagnetic stress-energy tensor effectively replaces the cosmological constant by minus the square of the topological mass and produces the stress-energy of a pressure-free perfect fluid. We show how a particular solution is related to the original Gödel universe and analyze the solutions from the point of view of identifications. Finally, we compute the conserved charges and work out the thermodynamics.

PACS numbers: 04.70.-s,04.40.Nr,11.30.-j,12.10.-g

I. INTRODUCTION

Exact solutions of higher dimensional gravity and supergravity theories play a key role in the development of string theory. Recently, a Gödel-like exact solution of five-dimensional minimal supergravity having the maximum number of supersymmetries has been constructed [1]. As its four dimensional predecessor, discovered by Gödel in 1949 [2], this solution possesses a large number of isometries. It can be lifted to higher dimensions and has recently been extensively studied as a background for string and M-theory, see e.g. [3, 4].

The Gödel-like five-dimensional solution found in [1] is supported by an Abelian gauge field. This gauge field has an additional Chern-Simons interaction and produces the stress-energy tensor of a pressureless perfect fluid. Since a Chern-Simons term can also be added in three dimensions, it is a natural question to ask whether a Gödel like solution exists in three-dimensional gravity coupled to a Maxwell-Chern-Simons field.

Actually, there is a stronger motivation to look for this kind of solutions of three-dimensional gravity. The reason is that the original four-dimensional Gödel spacetime is already effectively three dimensional, see e.g. [5]. In fact, the metric has as direct product structure $ds^2_{(4)} = ds^2_{(3)} + dz^2$ where $ds^2_{(3)}$ satisfies a purely three-dimensional Einstein equation.

The goal of this paper is twofold. On the one hand we will show that the three-dimensional factor $ds^2_{(3)}$ of the Gödel spacetime and its generalizations [6] are exact solutions of the three-dimensional Einstein-Maxwell-Chern-Simon theory described by the action [33],

$$I = \frac{1}{16\pi G}\int d^3x \left[\sqrt{-g}\left(R + \frac{2}{l^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) - \frac{\alpha}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\right]. \quad (1)$$

The stress-energy tensor of the perfect fluid will be fully generated by the gauge field A_μ, in complete analogy with the five-dimensional results reported in [1].

Our second goal deals with Gödel particles and black holes. Within the five dimensional supergravity theory, rotating black hole solutions on the Gödel background have been investigated in [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. It is then natural to ask whether the three-dimensional Gödel spacetime $ds^2_{(3)}$ can be generalized to include horizons. This is indeed the case and a general solution will be displayed [34].

Let us now briefly discuss the general structure of the stress-energy tensor of Maxwell-Chern-Simons theory. The original Gödel geometry is a solution of the Einstein equations in the presence of a pressureless fluid with energy density ρ and a negative cosmological constant Λ such that $\Lambda = -4\pi G\rho$. Equivalently, it can be viewed as a homogeneous spacetime filled with a stiff fluid, that is, $p_{SF} = \rho_{SF} = \rho/2$ and vanishing cosmological constant.

In (2+1)-spacetime dimensions, an electromagnetic field can be the source of such a fluid. To see this it is convenient to write the stress-energy tensor in terms of the dual field ${}^*F^\mu$,

$$16\pi G\, T^{\mu\nu} = {}^*F^{\mu}\,{}^*F^{\nu} - \frac{1}{2}{}^*F^{\alpha}\,{}^*F_{\alpha}g^{\mu\nu}\ . \quad (2)$$

In any region where the field ${}^*F^\mu$ is timelike, the electromagnetic field behaves as a stiff fluid with

$$u^\mu = \frac{1}{\sqrt{-{}^*F^{\alpha}\,{}^*F_\alpha}}{}^*F^\mu, \qquad \rho_{SF} = p_{SF} = -{}^*F^{\alpha}\,{}^*F_\alpha/2. \quad (3)$$

If Gödel's geometry is going to be a solution of the Einstein-Maxwell system, then $\rho_{SF} = -{}^*F^{\alpha}\,{}^*F_\alpha/2$ must be a constant. Moreover in comoving coordinates, in which $g_{tt} = -1$, ${}^*F^\mu$ must be a constant vector pointing

along the time coordinate. One can easily see that such a solution does not exist. In fact, the Maxwell equations for this solution,

$$d^*F = 0, \tag{4}$$

imply in these coordinates that $g_{t[\varphi,r]} = 0$ which cannot be achieved for Gödel. If the electromagnetic field acquires a topological mass α, however, Maxwell's equations (4) will be modified by the addition of the term αF. In that case, the timelike, constant, electromagnetic field is, as we will see below, a solution of the coupled Einstein-Maxwell-Chern-Simons system, and the geometry is precisely that of Gödel.

Finally, we compute the conserved charges - mass, angular momentum and electric charge - for these solutions and derive the first laws for the three dimensional black holes, adapted to an observer at rest with respect to the electromagnetic fluid. We then show how to adapt this first law in order to compare with the one for AdS black holes in the absence of the electromagnetic fluid.

II. GÖDEL SPACETIME AND TOPOLOGICALLY MASSIVE GRAVITO-ELECTRODYNAMICS

We start by reviewing the main properties, relevant to our discussion, of the four-dimensional Gödel spacetimes [2, 6, 17]. These metrics have a direct product structure $ds^2_{(3)} + dz^2$ with three-dimensional factor given by

$$\begin{aligned} ds^2_{(3)} &= -\left(dt + \frac{4\Omega}{\tilde{m}^2}\sinh^2\left(\frac{\tilde{m}\rho}{2}\right)d\varphi\right)^2 + d\rho^2 \\ &\quad + \frac{\sinh^2(\tilde{m}\rho)}{\tilde{m}^2}d\varphi^2. \end{aligned} \tag{5}$$

The original solution discovered by Gödel corresponds to $\tilde{m}^2 = 2\Omega^2$. Furthermore, it was pointed out in [6] that the property of homogeneity and the causal structure of the Gödel solution also hold for Ω and $\tilde{m}$ independent, provided that $0 \leq \tilde{m}^2 < 4\Omega^2$, the limiting case $\tilde{m}^2 = 4\Omega^2$ corresponding to anti-de Sitter space.

The three-dimensional metric (5) has 4 independent Killing vectors, two obvious ones, $\xi_{(1)} = \partial_t$ and $\xi_{(2)} = \partial_\varphi$, and two additional ones,

$$\begin{aligned} \xi_{(3)} &= \frac{2\Omega}{\tilde{m}^2}\tanh(\tilde{m}\rho/2)\sin\varphi\frac{\partial}{\partial t} - \frac{1}{\tilde{m}}\cos\varphi\frac{\partial}{\partial\rho} + \\ &\quad \coth(\tilde{m}\rho)\sin\varphi\frac{\partial}{\partial\varphi}, \end{aligned} \tag{6}$$

$$\begin{aligned} \xi_{(4)} &= \frac{2\Omega}{\tilde{m}^2}\tanh(\tilde{m}\rho/2)\cos\varphi\frac{\partial}{\partial t} + \frac{1}{\tilde{m}}\sin\varphi\frac{\partial}{\partial\rho} + \\ &\quad \coth(\tilde{m}\rho)\cos\varphi\frac{\partial}{\partial\varphi}. \end{aligned} \tag{7}$$

which span the algebra $so(2,1)\times\mathbb{R}$. Finally, the metric (5) satisfies the three dimensional Einstein equations,

$$G^{\mu\nu} - \Omega^2 g^{\mu\nu} = (4\Omega^2 - \tilde{m}^2)\delta^\mu_t\delta^\nu_t, \tag{8}$$

for all values of $\Omega, \tilde{m}$, and we see that Ω plays the role of a negative cosmological constant.

Note that a solution $ds^2_{(3)}$ to Einstein's equations in 3 dimensions can be lifted to a solution in 4 dimensions through the addition of a flat direction z if the additional components of the stress-energy tensor are chosen as $\mathcal{T}^{\mu z} = 0$ and $\mathcal{T}^{zz} = g_{\mu\nu}\mathcal{T}^{\mu\nu} + \Omega^2/4\pi G$. For the solutions (5), $\mathcal{T}^{zz} = (\tilde{m}^2 - 2\Omega^2)/8\pi G$ and vanishes, as it should, for the original Gödel solution.

Our first goal is to prove that (5) can be regarded as an exact solution to the equations of motion following from (1).

To this end, we need to supplement (5) with a suitable gauge field which will provide the stress-energy tensor (right hand side of (8)). Consider a spherically symmetric gauge field in the gauge $A_r = 0$,

$$A = A_t(\rho)dt + A_\varphi(\rho)d\varphi. \tag{9}$$

Inserting this ansatz for the gauge field into the equations of motion associated to the action (1), and assuming that the metric takes the form (5), one indeed finds a solution for A_t and A_φ. Moreover, the two parameters $\tilde{m}, \Omega$ entering in (5) become related to the coupling constants α and $1/l$ as

$$\begin{aligned} \Omega &= \alpha, \\ \tilde{m}^2 &= 2\left(\alpha^2 + \frac{1}{l^2}\right). \end{aligned} \tag{10}$$

With this parameterization, the Gödel sector is determined by $\alpha^2 l^2 - 1 > 0$, with $\alpha^2 l^2 = 1$ corresponding to anti-de Sitter space. For future convenience, we shall write the solution in terms of a new radial coordinate r defined by

$$r = \frac{2}{\tilde{m}^2}\sinh^2\left(\frac{\tilde{m}\rho}{2}\right). \tag{11}$$

Explicitly, the metric and gauge field are given by,

$$\begin{aligned} ds^2 &= -dt^2 - 4\alpha r dt d\varphi + \left[2r - \left(\alpha^2 l^2 - 1\right)\frac{2r^2}{l^2}\right]d\varphi^2 \\ &\quad + \left(2r + (\alpha^2 l^2 + 1)\frac{2r^2}{l^2}\right)^{-1} dr^2, \end{aligned} \tag{12}$$

$$A = \sqrt{\alpha^2 l^2 - 1}\,\frac{2r}{l}\,d\varphi. \tag{13}$$

From now on, we always write Ω and $\tilde{m}$ in terms of α and l using (10). The general solution for A involves the addition of arbitrary constant terms along dt and $d\varphi$ in (13). At this stage, we choose the constant in A_t to be zero. We will come back to this issue when we discuss black hole solutions below. A constant term in A_φ is not allowed, however, if one requires $A_\varphi d\varphi$ to be regular everywhere. Indeed, near $r = 0$, the spacelike surfaces of (12) are $\mathbb{R}^2$ in polar coordinates, the radial coordinate r in (12) being the square root of a standard

radial coordinate over $\mathbb{R}^2$, and thus A_φ must vanish at $r=0$ because the 1-form $d\varphi$ is not well defined there.

The gauge field (13) is also invariant under the isometries of (5), up to suitable gauge transformations: for each Killing vector $\xi^\mu_{(a)}$ there exists a function $\epsilon_{(a)}$ such that

$$\mathcal{L}_{\xi_{(a)}} A_\mu - \partial_\mu \epsilon_{(a)} = 0. \tag{14}$$

In this sense, the Killing vectors $\xi^\mu_{(a)}$ of (5) are lifted to gauge parameters $(\xi^\mu_{(a)}, \epsilon_{(a)})$ that leave the full gravity plus gauge field solution invariant. The generalized Gödel metric (12) together with the gauge field (13) define a background for the action (1) with 4 linearly independent symmetries of this type. We shall now use these symmetries in order to find new solutions describing particles and black holes (see also [18]).

III. GÖDEL PARTICLES: $\alpha^2 l^2 > 1$

We have proven in the previous section that the Gödel metric can be regarded as an exact solution to action (1). The associated gauge field (13) is however real only in the range $\alpha^2 l^2 \geq 1$. We consider in this section the case $\alpha^2 l^2 > 1$ and introduce particle-like objects on the background (12) by means of spacetime identifications.

A. Gödel Cosmons

Identifications in three-dimensional gravity were first introduced by Deser, Jackiw and t'Hooft [19, 20] and the resulting objects called "cosmons". In the presence of a topologically massive electromagnetic field, cosmons living in a Gödel background may also be constructed along these lines.

Take the metric (12) and make the following identification along the Killing vectors ∂_φ and ∂_t

$$(t, \varphi) \sim (t - 2\pi j m, \varphi + 2\pi m).$$

where m, j are real constants. If $m \neq 1$ this procedure will turn the spatial plane into a cone. The cosmon lives in the tip of this cone, and its mass is related to m and j (see below). The time-helical structure given by j will provide angular momentum.

To analyze the resulting geometry it is convenient to pass to a different set of coordinates,

$$\begin{aligned} \varphi &= \varphi' m \\ t &= t' - j\varphi' m \\ r &= \frac{r'}{m} + \frac{j}{2\alpha}. \end{aligned} \tag{15}$$

where the above identification amounts to

$$\varphi' \sim \varphi' + 2\pi n, \qquad n \in Z. \tag{16}$$

Also, the new time t' flows ahead smoothly, that is, it does not jump after encircling the particle. Inserting these coordinates into (12), and erasing the primes, we find the new metric

$$\begin{aligned} ds^2 &= -dt^2 - 4\alpha r dt d\varphi \\ &+ \left[8G\nu r - (\alpha^2 l^2 - 1)\frac{2r^2}{l^2} - \frac{4GJ}{\alpha}\right] d\varphi^2 \\ &+ \left((\alpha^2 l^2 + 1)\frac{2r^2}{l^2} + 8G\nu r - \frac{4GJ}{\alpha}\right)^{-1} dr^2. \end{aligned} \tag{17}$$

For fixed m and mj, the new constants ν and J are given by

$$4G\nu = m\left(1 + \frac{1+\alpha^2 l^2}{\alpha l^2} j\right), \tag{18}$$

$$4GJ = -m^2 j\left(1 + \frac{1+\alpha^2 l^2}{2\alpha l^2} j\right). \tag{19}$$

These constants will be shown to be related to the mass and angular momentum respectively.

Since under (15) φ scales with m while r with $1/m$ we see that the r-dependent part of gauge field (13) is invariant under (15). However, the manifold now has a non-trivial cycle, and it is not regular at the point $r = r_0$ invariant under the action of the Killing vector whose orbits are used for identifications. Explicitly, $r_0 = -\frac{jm}{2\alpha}$ which corresponds to $r = 0$ before the shift of r in (15). This means that one can now add a constant piece to A_φ. The new gauge field becomes

$$A = \left(-\frac{4GQ}{\alpha} + \sqrt{\alpha^2 l^2 - 1}\,\frac{2r}{l}\right) d\varphi. \tag{20}$$

The constant Q will be identified below as the electric charge of the particle sitting at $r = 0$.

The metrics (17) only admit the 2 Killing vectors ∂_t and ∂_φ. Indeed, the other candidates $\xi_{(3)}$ and $\xi_{(4)}$ do not survive as they do not commute with the Killing vector along which the identifications are made [21].

So far we have only used the Killing vectors $\partial/\partial\varphi$ and $\partial/\partial t$ of (5) to make identifications. Besides these Killing vectors, the metric (5) has two other isometries defined by the vectors (6) and (7), and one may consider identifications along them. We shall not explore this possibility in this paper.

B. Horizons, Singularities and Time Machines

Distinguished places of the geometry (49) may appear on those points where either $g_{\varphi\varphi}$ or g^{rr} vanishes. The vanishing of $g_{\varphi\varphi}$ indicates that $g_{\varphi\varphi}$ changes sign and hence closed timelike curves (CTC) appear. On the other hand, the vanishing of g^{rr} indicates the presence of horizons, as can readily be seen by writting (17) in ADM form.

The function $g_{\varphi\varphi}$ in (17) is an inverted parabola, and, it will have two zeros, say r_1 and r_2 whenever

$$2G\nu^2 > \frac{J(\alpha^2 l^2 - 1)}{\alpha l^2}. \tag{21}$$

We must require this condition to be fulfilled in order to have a "normal" region where ∂_φ is spacelike. The boundary of the normal region are two spacelike surfaces, the velocity of light surfaces (VLS) at $r = r_1$ and $r = r_2$ (assume $r_2 > r_1$). These surfaces are perfectly regular as long as $g_{t\varphi} \neq 0$ there, which is indeed the case for the metric (17), when $\alpha \neq 0$.

On the other hand, it is direct to see from (17) that

$$g^{rr} = 4\alpha^2 r^2 + g_{\varphi\varphi}. \tag{22}$$

Since $g_{\varphi\varphi}$ is positive in the normal region, there are no horizons there and g^{rr} is positive in that region. This together with the fact that g^{rr} is the parabola of Fig. 1, means that, if any, both zeros of g^{rr} are on the same side of the normal region. The sides in which no zero of g^{rr} are present are analog to the Gödel time machine, an unbounded region, free of singularities, where ∂_φ is timelike. If $\nu \geq 0$, the roots of g^{rr} are smaller than the roots of $g_{\varphi\varphi}$. Without loss of generality, we can restrict ourselves to this case because the solutions parametrized by (ν, J, Q) are related to those with $(-\nu, J, -Q)$ by the change of coordinates $r \to -r$, $\varphi \to -\varphi$.

The condition for "would be horizons" is

$$2G\nu^2 > \frac{J(\alpha^2 l^2 + 1)}{\alpha l^2}. \tag{23}$$

As depicted in Fig. 1, once one reaches the largest root $r_+ = r_0$ of g^{rr}, the manifold comes to an end. Indeed, the signature of the metric changes as one passes $g^{rr} = 0$. This can be seen by putting the metric in ADM form (see (50) below). Note that in this case, given (ν, J), there is a unique (m, mj) satisfying (18)-(19).

Using then $r = r_+ + \kappa_0 |\alpha r_+| \rho^2$, with r_- the smallest root of g^{rr} and $\kappa_0 = \frac{(r_+ - r_-)(\alpha^2 l^2 + 1)}{2l^2 |\alpha r_+|}$, one finds near r_+,

$$ds^2 \approx \kappa_0^2 \rho^2 dt^2 + d\rho^2 - 4\alpha^2 r_+^2 (d\varphi + \frac{dt}{2\alpha r_+})^2. \tag{24}$$

This means that the spacetime has a naked singularity at r_+, which is the analog of the one found in the spinning cosmon of [19, 20].

Alternatively, as proposed originally in [15] for the case where the would be horizon is inside the time machine, one can periodically identify time t with period $2\pi/\kappa_0$. This leads to having CTC's lying everywhere, including the normal region.

IV. GÖDEL BLACK HOLES

A. The $\alpha^2 l^2 < 1$ sector

We have shown in Sec. II that the metric (5) can be embedded as an exact solution to the equations of motion

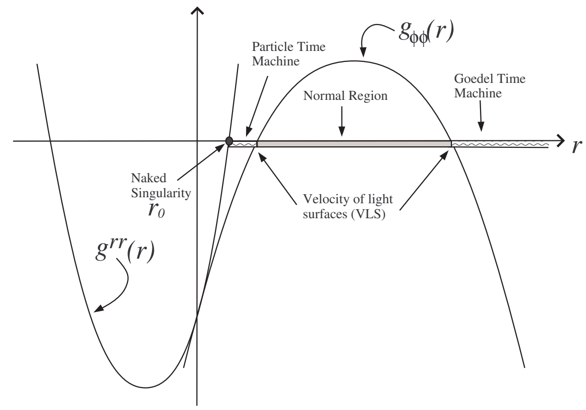

FIG. 1: Gödel cosmons

derived from (1). The necessary gauge field, given in (13) is, however, real only in the range $\alpha^2 l^2 \geq 1$. As we mentioned in Sec. II, the gauge field (13) represents the most general static spherically symmetric solution, given the metric (5) (or, in the new radial coordinate, (12)). This means that if we want to find a real gauge field in the range $\alpha^2 l^2 < 1$ we need to start with a different metric. The goal of this section is to explore the other sector, $\alpha^2 l^2 < 1$, where black holes will be constructed.

Starting from the metric (12) and gauge field (13) it is easy to construct a new exact solution which will be real in the range $\alpha^2 l^2 < 1$. Consider the following (complex) coordinate changes [35] acting on (12) and (13): $\varphi \to i\varphi$, $t \to it$, and $r \to -r$. The new metric and gauge field read,

$$\begin{aligned} ds^2 &= dt^2 - 4\alpha r dt d\varphi + \left[2r - \left(1 - \alpha^2 l^2\right) \frac{2r^2}{l^2}\right] d\varphi^2 \\ &\quad + \left((\alpha^2 l^2 + 1)\frac{2r^2}{l^2} - 2r\right)^{-1} dr^2 \end{aligned} \tag{25}$$

$$A = \sqrt{1 - \alpha^2 l^2}\, \frac{2r}{l} d\varphi. \tag{26}$$

Several comments are in order here. First of all, the intermediate step of making some coordinates complex is only a way to find a new solution. From now on, all coordinates t, r, φ are defined real, and, in that sense, the fields (25) and (26) provide a new exact solution to the action (1) which is real in the range $\alpha^2 l^2 < 1$.

Second, in the original metric (12), the coordinate φ was constrained by the geometry to have the range $0 \leq \varphi < 2\pi$. This is no longer the case in the metric (25). The 2-dimensional sub-manifold described by the coordinates r, φ does not have the geometry of $\mathbb{R}^2$ near $r \to 0$ anymore; the coordinate φ is thus not constrained to be compact, and in principle it should have the full range

$$-\infty < \varphi < \infty. \tag{27}$$

The reason that φ in (25) is not constrained by the geometry is that the g^{rr} component of the metric (25) changes sign as we approach $r = 0$. This is an indication of the presence of a horizon, although this surface is not yet compact.

Finally, it is worth mentioning that the metrics (25) and (12) are real and are related by a coordinate transformation, so that all local invariants involving the metric alone have the same values. However, as solutions to the Einstein-Maxwell equations, they are inequivalent. Indeed, the diffeomorphism and gauge invariant quantity $({}^*F)^2 = 4(1-\alpha^2 l^2)/l^2$ changes sign when going from (12)-(13) to (25)-(26). This is different from the pure anti-de Sitter case where particles and black holes are obtained by identifications performed on the same background.

B. The Gödel black hole

Let us go back to (25) and note that the function g^{rr} vanishes at $r_+ > 0$. In order to make the $r = r_+$ surface a regular, finite area, horizon we shall use the Killing vector ∂_φ of (25) to identify points along the φ coordinate. In this case, ∂_φ has a non-compact orbit and identifications along it does not produce a conical singularity, but a “cylinder”. More generically, we may proceed in analogy with the cosmon case and identify along a combination of both ∂_φ and ∂_t so that

$$(t, \varphi) \sim (t - 2\pi jm, \varphi + 2\pi m).$$

so that the resulting geometry will also carry angular momentum. We again pass to a different set of coordinates,

$$\begin{aligned} \varphi &= \varphi' m && (28)\\ t &= t' - j\varphi' m && (29)\\ r &= \frac{r'}{m} - \frac{j}{2\alpha}, && (30) \end{aligned}$$

so that the new angular coordinate φ' is identified in 2π, and the time t' flows ahead smoothly.

The new metric reads (after erasing the primes),

$$ds^2 = dt^2 - 4\alpha r dt d\varphi + \left(8G\nu r - (1-\alpha^2 l^2)\frac{2r^2}{l^2} - \frac{4GJ}{\alpha}\right) d\varphi^2 + \left((\alpha^2 l^2 + 1)\frac{2r^2}{l^2} - 8G\nu r + \frac{4GJ}{\alpha}\right)^{-1} dr^2. \quad (31)$$

As for the particles analyzed in the previous section, for given (m, mj), we define new constants μ and J according to

$$4G\nu = m\left(1 + \frac{1+\alpha^2 l^2}{\alpha l^2} j\right), \quad (32)$$

$$4GJ = m^2 j\left(1 + \frac{1+\alpha^2 l^2}{2\alpha l^2} j\right). \quad (33)$$

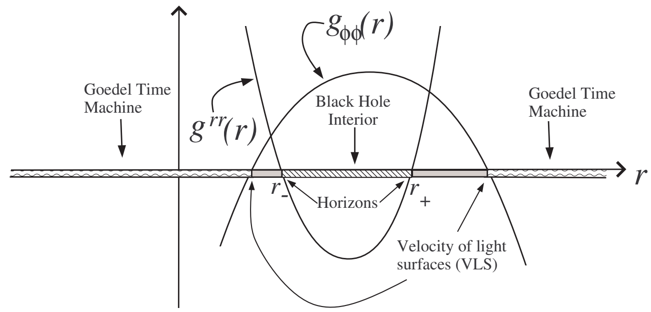

FIG. 2: Gödel black holes

Again, these constants will be related below to the mass and the angular momentum and without loss of generality, we can limit ourselves to the case $\nu \geq 0$.

In the new coordinates, the electromagnetic potential takes the form $A = A_\varphi d\varphi$, where

$$A_\varphi(r) = -\frac{4GQ}{\alpha} + \sqrt{1-\alpha^2 l^2}\,\frac{2r}{l}. \quad (34)$$

The constant Q is arbitrary because, once again, the nontrivial topology allows the addition of an arbitrary constant in A_φ. It is worth stressing that if φ was not compact, then m and Q would be trivial constants. It also follows that the Killing vectors of (25) have the same form as those of (12), but with the trigonometric functions $\cos(\varphi)$ and $\sin(\varphi)$ replaced by hyperbolic ones. Again, these vectors do not survive after the identifications.

C. Horizons, Singularities and Time Machines

We now proceed to analyze the metric in the same way we did in the preceding section. Again we have a condition for having a normal region, which, in this case reads

$$2G\nu^2 > \frac{J(1-\alpha^2 l^2)}{\alpha l^2}. \quad (35)$$

The functions g^{rr} and $g_{\varphi\varphi}$ now behave as in Fig. 2.

Note that

$$g^{rr} = -g_{\varphi\varphi} + 4\alpha^2 r^2, \quad (36)$$

and therefore horizons may only exist in the normal region of positive $g_{\varphi\varphi}$. Note, however, that for horizons to exist we must require

$$2G\nu^2 \geq \frac{J(1+\alpha^2 l^2)}{\alpha l^2}. \tag{37}$$

If this requirement is fulfilled, we get two horizons inside the normal region, $r_- = mj/(2\alpha)$ and r_+, which coincide in the extremal case. The whole normal region is in fact an ergoregion because $\partial/\partial t$ is spacelike everywhere. Again, for given (ν, J), one can then find a unique solution (m, mj) satisfying (33)-(33).

Following Carter [23], the metric and the gauge field can be made regular at both horizons by a combined coordinate and gauge transformation. Indeed, if

$$\Delta(r) = (\alpha^2 l^2 + 1)\frac{2r^2}{l^2} - 8G\mu r + \frac{4GJ}{\alpha},$$

the black hole metric can be written as

$$ds^2 = (dt - 2\alpha r d\varphi)^2 - \Delta d\varphi^2 + \frac{dr^2}{\Delta}. \tag{38}$$

The analog of ingoing Eddington-Finkelstein coordinates are the angle $\overleftarrow{\varphi}$ and the time $\overleftarrow{t}$ defined by $d\varphi = d\overleftarrow{\varphi} - \frac{1}{\Delta}dr$, $dt = d\overleftarrow{t} - \frac{2\alpha r}{\Delta}dr$, giving the regular metric

$$ds^2 = (d\overleftarrow{t} - 2\alpha r d\overleftarrow{\varphi})^2 - \Delta d\overleftarrow{\varphi}^2 + 2d\overleftarrow{\varphi}\, dr. \tag{39}$$

With $A_\varphi(r)$ given by (34), the r dependent gauge transformation $\overleftarrow{A} = A + d\epsilon$, where $\epsilon = \int dr \frac{A_\varphi(r)}{\Delta}$ gives the regular potential $\overleftarrow{A} = A_\varphi(r)d\overleftarrow{\varphi}$ whose norm $\overleftarrow{A}^2$ is zero.

Outgoing Eddington-Finkelstein coordinates are defined by $d\varphi = -d\overrightarrow{\varphi} + \frac{1}{\Delta}dr$, $dt = -d\overrightarrow{t} + \frac{2\alpha r}{\Delta}dr$. The metric then takes also the form (39) with $\overleftarrow{t}$ and $\overleftarrow{\varphi}$ replaced by $-\overrightarrow{t}$ and $-\overrightarrow{\varphi}$ and the potential can be regularized by $\overrightarrow{A} = A - d\epsilon$.

The null generators of the horizons are $\frac{\partial}{\partial t} + \frac{1}{2\alpha r_\pm}\frac{\partial}{\partial \varphi}$. The associated ignorable coordinates which are constant on these null generators are then given by

$$dt^\pm = dt - 2\alpha r_\pm d\varphi. \tag{40}$$

Kruskal type coordinates $(t^\pm, U^\pm, V^\pm)$ are obtained by defining

$$k_\pm \frac{dV^\pm}{V^\pm} \;=\; d\overleftarrow{\varphi} = d\varphi + \frac{dr}{\Delta}, \tag{41}$$

$$k_\pm \frac{dU^\pm}{U^\pm} \;=\; d\overrightarrow{\varphi} = -d\varphi + \frac{dr}{\Delta}, \tag{42}$$

where

$$k_\pm = \frac{l^2}{1+\alpha^2 l^2}\frac{1}{r_\pm - r_\mp}.$$

In these coordinates, the metric is manifestly regular at the bifurcation surfaces,

$$\begin{aligned} ds^2 \;&=\; \left[dt^\pm - \alpha k_\pm (r - r_\mp)(U^\pm dV^\pm - V^\pm dU^\pm)\right]^2 \\ &+\; \frac{2k_\pm (r-r_\mp)^2}{r_\pm - r_\mp} dU^\pm dV^\pm, \end{aligned} \tag{43}$$

with r given implicitly by

$$U^\pm V^\pm = \left(\frac{r - r_+}{r - r_-}\right)^{\pm 1}. \tag{44}$$

In Kruskal coordinates, the gauge field (34) becomes

$$\begin{aligned} A \;&=\; \frac{k_\pm}{2}\left(\frac{A_\varphi(r_\pm)}{U^\pm V^\pm} + \frac{\sqrt{1-\alpha^2 l^2}}{l}(r - r_\mp)\right) \\ &\times (U^\pm dV^\pm - V^\pm dU^\pm). \end{aligned} \tag{45}$$

The potential can be regularized at $r = r_\pm$ by the transformations

$$\begin{aligned} \tilde{A}^\pm \;&=\; A - d[A_\varphi(r_\pm)\frac{k_\pm}{2}\ln\frac{V^\pm}{U^\pm}] \\ &=\; \frac{k_\pm \sqrt{1-\alpha^2 l^2}}{2l}(r - r_\mp)(U^\pm dV^\pm - V^\pm dU^\pm). \end{aligned} \tag{46}$$

In the original coordinates, however, the parameters of these transformations explicitly involve the angle φ, $\tilde{A}^\pm = A - d[A_\varphi(r_\pm)\varphi]$ and, as explicitly shown below, they change the electric charge. In order to avoid this, one can add a constant piece proportional to $dt^\pm$, so that

$$A^\pm \;=\; \tilde{A}^\pm - d(\frac{A_\varphi(r_\pm)}{2\alpha r_\pm}t^\pm). \tag{47}$$

In the original coordinates, the gauge parameter is now a linear function of t alone,

$$A^\pm \;=\; A - d(\frac{A_\varphi(r_\pm)}{2\alpha r_\pm}t). \tag{48}$$

According to the definition given below, such a transformation does not change the charges.

The causal structure of the Gödel black hole is displayed in the Carter-Penrose diagram Fig. 3, where each point represents a circle.

V. VACUUM SOLUTIONS $\alpha^2 l^2 = 1$

In the case $\alpha^2 l^2 = 1$ the gauge field vanishes and the Gödel metric (12) reduces to the three-dimensional anti-de Sitter space (to see this, do the coordinate transformations $\varphi \to \varphi + \alpha t$ and $2r \to r^2$). This means that the identifications in this case yield the usual three-dimensional black holes, and conical singularities.

VI. THE GENERAL SOLUTION

A. Reduced equations of motion

We have seen in previous sections that the Gödel metrics (5) and (25), as well as the corresponding quotient spaces describing particles and black holes, can be regarded as exact solutions to the action (1).

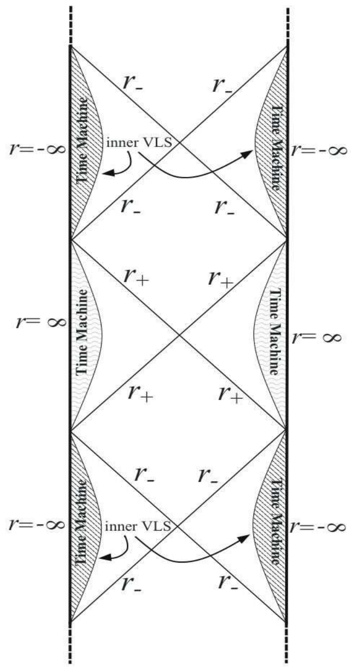

FIG. 3: Carter-Penrose diagram of Gödel black hole

We have distinguished three cases according to the values of the dimensionless quantity $\alpha^2 l^2$. Our purpose in this section is to write a general solution which will be valid for all values of $\alpha^2 l^2$. We shall now construct the solution by looking directly at the equations of motion. It is useful to write a general spherically symmetric static ansatz in the form [24, 25]

$$ds^2 = \frac{dr^2}{h^2 - p\,q} + p\,dt^2 + 2h\,dt d\varphi + q\,d\varphi^2, \quad (49)$$

where p, q, h are functions of r only. This ansatz can also be written in the "ADM form",

$$ds^2 = -\frac{h^2 - pq}{q}dt^2 + \frac{dr^2}{h^2 - pq} + q\left(d\varphi + \frac{h}{q}dt\right)^2. \quad (50)$$

This confirms that the function g^{rr}

$$f(r) = h^2(r) - p(r)q(r), \quad (51)$$

controls the existence of horizons. Note that for all p, q, h, the determinant of this metric is $\det(-g) = 1$. For the gauge field, we use the radial gauge $A_r = 0$, and assume that A_t and A_φ depend only on the radial coordinate,

$$A = A_t(r)\,dt + A_\varphi(r)\,d\varphi. \quad (52)$$

In this parametrization, Einstein's equations take the remarkably simple form,

$$\begin{aligned} h'' &= -A_t'\,A_\varphi' \\ p'' &= -A_t'^2 \\ q'' &= -A_\varphi'^2 \\ (h^2 - pq)'' &= h'^2 - p'q' + \frac{4}{l^2}, \end{aligned} \quad (53)$$

where primes denote radial derivatives. Maxwell's equations reduce to

$$\begin{aligned} &(hA_t' - pA_\varphi' - 2\alpha A_t)' = 0, \\ &(qA_t' - hA_\varphi' - 2\alpha A_\varphi)' = 0. \end{aligned} \quad (54)$$

Before we write the solution to these equations, we make some general remarks on the structure of the stress-energy tensor associated to topologically massive electrodynamics. As we pointed out in the introduction, we will seek for solutions with a constant electromagnetic field ${}^*F$. Hence, we will only consider potentials A which are linear in r. In this case, Eqs. (54) are

$$\begin{aligned} h'A_t' - p'A_\varphi' &= 2\alpha A_t', \\ q'A_t' - h'A_\varphi' &= 2\alpha A_\varphi'. \end{aligned} \quad (55)$$

We now multiply the first by h' and the second by p', then we subtract them to obtain

$$(h'^2 - p'q')A_t' = 2\alpha(h'A_t' - p'A_\varphi') = 4\alpha^2 A_t'.$$

In the last step we have used Eq. (55). This implies that, if $A_t' \neq 0$ then $(h'^2 - p'q') = 4\alpha^2$. By properly manipulating Eqs. (55) we see that this result is also valid if $A_t' = 0$ but $A_\varphi' \neq 0$, and therefore is it true as long as the electromagnetic field does not vanish. Now we insert this in the last equation in (53), and obtain,

$$\begin{aligned} {}^*F^{\mu\,*}F_\mu &= q(A_t')^2 + p(A_\varphi')^2 - 2hA_t'\,A_\varphi' \\ &= \frac{4}{l^2}\left(1 - \alpha^2 l^2\right). \end{aligned} \quad (56)$$

This equation tells us that when the topological mass α^2 is greater (smaller) than the negative cosmological constant $1/l^2$, the theory only supports timelike (spacelike) constants fields. Therefore, for the generalized Gödel spacetimes (5), we will need a topological mass $\alpha^2 > 1/l^2$. In the other region, the constant electromagnetic field will describe a tachyonic perfect fluid. Anyway, as we will see below, it is this region in which black hole solutions are going to exist.

B. The solution

By direct computation one can check that equations (53)-(54) are satisfied by the field

$$
\begin{aligned}
p(r) &= 8G\mu \\
q(r) &= -\frac{4GJ}{\alpha} + 2r - 2\frac{\gamma^2}{l^2}r^2 \\
h(r) &= -2\alpha r \\
A_t(r) &= \frac{\alpha^2 l^2 - 1}{\gamma \alpha l} + \zeta \\
A_\varphi(r) &= -\frac{4G}{\alpha}Q + 2\frac{\gamma}{l}r,
\end{aligned}
\tag{57}
$$

where

$$
\gamma = \sqrt{\frac{1-\alpha^2 l^2}{8G\mu}}. \tag{58}
$$

The parameters μ, J and Q are integration constants with a physical interpretation as they will be identified with mass, angular momentum and electric charge below. The arbitrary constant ζ on the other hand will be shown to be pure gauge. For later convenience, it is however useful to keep it along and not restrict ourselves to a particular gauge at this stage. This will be discussed in details Sec. VII.

In the sector $\alpha^2 l^2 > 1$, the solution is real only for μ negative. These are the Gödel particles, i.e., the conical singularities, discussed in Sec. III. The metric (17) is recovered when $\mu = -2G\nu^2$ and the change of variables $t \to t/\sqrt{-8G\mu}$, $r \to \sqrt{-8G\mu}\, r$ is performed. For the special values $\mu = -1/8G$ and $J = 0$, which correspond to the trivial identification $j = 0$, $m = 1$ in Sec. III, the conical singularities disappear and we are left with the Gödel universes (12), used for the identifications producing the cosmons.

When $\alpha^2 l^2 < 1$, μ has to be positive. The black hole metrics (31) of Sec. IV are recovered when $\mu = 2G\nu^2$ and $t \to t/\sqrt{8G\mu}$, $r \to \sqrt{8G\mu}\, r$. For $\mu = 1/8G$ and $J = 0$, they reduce to the solution (25) from which the black holes have been obtained from non-trivial identifications.

By construction, the electromagnetic stress-energy tensor for the solutions (57) takes the form

$$
8\pi G T^{\mu\nu}_{EM} = (\alpha^2 - \frac{1}{l^2})g^{\mu\nu} + 8\pi G \mathcal{T}^{\mu\nu}, \tag{59}
$$

$$
\mathcal{T}^{\mu\nu} = \frac{|1-\alpha^2 l^2|}{4\pi G l^2} u^\mu u^\nu, \tag{60}
$$

where the unit tangent vector of the fluid is $u = \frac{1}{\sqrt{8G|\mu|}}\frac{\partial}{\partial t}$. For $\alpha^2 l^2 \neq 1$, the effect of the electromagnetic field can be taken into account by replacing the original cosmological constant $-\frac{1}{l^2}$ by the effective cosmological constant $-\alpha^2$ and introducing a pressure-free perfect, ordinary or tachyonic, fluid with energy density $\frac{|1-\alpha^2 l^2|}{4\pi G l^2}$. From this point of view, *the Chern-Simons coupling transmutes into a cosmological constant.* For $1 - \alpha^2 l^2 < 0$, the fluid flows along timelike curves while for $1 - \alpha^2 l^2 > 0$, the fluid is tachyonic.

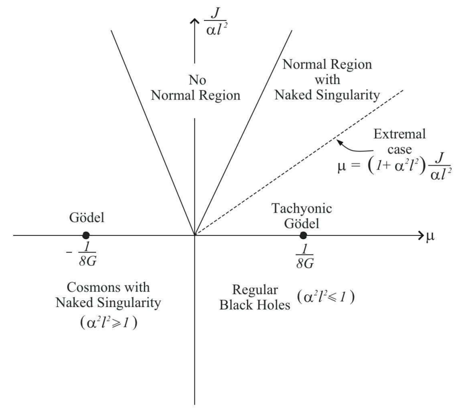

FIG. 4: Sectors of the general solution.

When $\alpha^2 l^2 = 1$, the fluid disappears, the stress-energy tensor vanishes and the solution is real for $\mu \in \mathbb{R}$. The metric (57) reduces to the BTZ metric [22], as can be explicitly seen by transforming to the standard frame that is non-rotating at infinity with respect to anti-de Sitter space,

$$
\varphi \to \varphi + \alpha t, \qquad r \to \frac{r^2}{2} + \frac{2GJ}{\alpha}. \tag{61}
$$

As will be explained in more details below, in the rotating frame that we have used, the energy and angular momentum are μ and J respectively, while they become $M \equiv \mu - \alpha J$ and J in the standard non-rotating frame.

Regular black holes have the range (see Fig. 5)

$$
\mu \geq 0, \qquad \mu \geq 2\alpha J. \tag{62}
$$

Note that the solution still possess a topological charge Q. It has been discussed in more details in [25].

When $\alpha^2 l^2 \neq 1$, the limit $\mu \to 0$ can be taken smoothly in the coordinates $\hat{r} = \gamma r$, $\hat{t} = t/\gamma$ in which the solution becomes

$$
\begin{aligned}
p(\hat{r}) &= 1 - \alpha^2 l^2 \\
q(\hat{r}) &= -\frac{4GJ}{\alpha} + \frac{2}{\gamma}\hat{r} - \frac{2}{l^2}\hat{r}^2 \\
h(\hat{r}) &= -2\alpha\hat{r} \\
A_{\hat{t}}(\hat{r}) &= \frac{\alpha^2 l^2 - 1}{\alpha l} + \hat{\zeta} \\
A_\varphi(\hat{r}) &= -\frac{4G}{\alpha}Q + \frac{2}{l}\hat{r},
\end{aligned}
\tag{63}
$$

where $\hat{\zeta} = \gamma\zeta$.

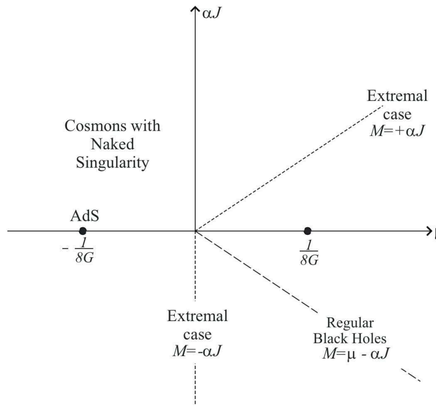

FIG. 5: Sectors of the $\alpha^2 l^2 = 1$ solution. The BTZ mass axis $M = \mu - \alpha J$ and the extremal solutions are explicitly indicated.

VII. CONSERVED CHARGES

A. Angular momentum, electric charge and energies

The charge differences between a given solution $(g_{\mu\nu}, A_\mu)$ and an infinitesimally close one $(g_{\mu\nu} + \delta g_{\mu\nu}, A_\mu + \delta A_\mu)$ are controlled by the linearized theory around $(g_{\mu\nu}, A_\mu)$. The equivalence classes of conserved $n-2$-forms (here 1-forms) of the linearized theory can be shown [26, 27] to be in one-to-one correspondence with gauge parameters (ξ^μ, ϵ) satisfying

$$\begin{cases} \mathcal{L}_\xi g_{\mu\nu} = 0, \\ \mathcal{L}_\xi A_\mu + \partial_\mu \epsilon = 0. \end{cases} \tag{64}$$

The associated on-shell closed 1-forms can be written as [14] (see also [28, 29, 30] for the gravitational part):

$$k_{\xi,\epsilon}[(\delta g, \delta A); (g, A)] = k^{grav}_\xi + k^{em}_{\xi,\epsilon} + k^{CS}_{\xi,\epsilon}, \tag{65}$$

where

$$k^{grav}_\xi = -\delta K^K_\xi + K^K_{\delta\xi} - \xi \cdot \Theta, \tag{66}$$

where

$$K^K_\xi = dx^\rho \frac{\sqrt{-g}}{16\pi G} \epsilon_{\rho\mu\nu} D^\mu \xi^\nu, \tag{67}$$

is the Komar 1-form and

$$\xi \cdot \Theta = dx^\rho \frac{\sqrt{-g}}{16\pi G} \epsilon_{\rho\nu\mu} \xi^\mu (g^{\nu\alpha} D^\beta \delta g_{\alpha\beta} - g^{\alpha\beta} D^\nu \delta g_{\alpha\beta}).$$

The electromagnetic contribution is

$$k^{em}_{\xi,\epsilon} = -\delta Q^{em}_{\xi,\epsilon} + Q^{em}_{\delta\xi,\delta\epsilon} - \xi \cdot \Theta^{em}, \tag{68}$$

where

$$Q^{em}_{\xi,\epsilon} = dx^\rho \epsilon_{\rho\mu\nu} \frac{\sqrt{-g}}{32\pi G} \left(F^{\mu\nu} (\xi^\rho A_\rho + \epsilon) \right), \tag{69}$$

$$\xi \cdot \Theta^{em} = dx^\rho \epsilon_{\rho\nu\mu} \frac{\sqrt{-g}}{16\pi G} \xi^\mu F^{\alpha\nu} \delta A_\alpha. \tag{70}$$

The Chern-Simons term contributes as

$$k^{CS}_{\xi,\epsilon} = dx^\rho \alpha \frac{\sqrt{-g}}{8\pi G} \delta A_\rho (A_\sigma \xi^\sigma + \epsilon). \tag{71}$$

Finite charge differences are computed by choosing a path γ in parameter space joining the solution (g, A) to a background solution $(\bar{g}, \bar{A})$ as

$$\mathcal{Q}_{\bar{\xi},\bar{\epsilon}} = \oint_S \int_\gamma k_{\xi,\epsilon}[(\delta g_\gamma, \delta A_\gamma); (g_\gamma, A_\gamma)], \tag{72}$$

where S is a closed 1-dimensional submanifold and $(\delta g_\gamma, \delta A_\gamma)$ denotes the directional derivative of the fields along γ in the space of parameters. These charges only depend on the homology class of S. They also do not depend on the path, provided the integrability conditions [31] $d_V \oint_S k_{\xi,\epsilon}[(d_V g, d_V A); (g, A)] = 0$ are satisfied.

For generic metrics and gauge fields of the form (57), the general solution (ξ, ϵ) of (64) is a linear combination of $(0, -1)$, $(-\frac{\partial}{\partial\varphi}, 0)$ and $(\frac{\partial}{\partial t}, 0)$. These basis elements are associated to infinitesimal charges as follows,

$$\begin{aligned} \oint_S k_{0,-1} &= \delta Q, \ \oint_S k_{-\frac{\partial}{\partial\varphi},0} = \delta(J - \frac{2G}{\alpha} Q^2), \\ \oint_S k_{\frac{\partial}{\partial t},0} &= \delta\mu - \zeta \delta Q, \end{aligned} \tag{73}$$

where the contribution proportional to δQ in $\oint_S k_{-\frac{\partial}{\partial\varphi},0}$ and $\oint_S k_{\frac{\partial}{\partial t},0}$ originate from the Chern-Simons term through (71). The conserved charges associated with $(0, -1)$, $(-\frac{\partial}{\partial\varphi}, 0)$ are thus manifestly integrable. We choose to associate the angular momentum to $(-\frac{\partial}{\partial\varphi}, -\frac{4GQ}{\alpha})$ so that its value be algebraically independent of Q. If one takes as basis element $(\frac{\partial}{\partial t}, -\zeta)$ instead of $(\frac{\partial}{\partial t}, 0)$, one gets a third integrable conserved charge equal to $\delta\mu$.

The integrated charges computed with respect to the background $\mu = 0 = J = Q$ and associated to $(\frac{\partial}{\partial t}, -\zeta)$, $(-\frac{\partial}{\partial\varphi}, -\frac{4GQ}{\alpha})$ and $(0, -1)$ are the mass, the angular momentum and the total electric charge respectively,

$$\mathcal{E} = \mu, \quad \mathcal{J} = J, \quad \mathcal{Q} = Q. \tag{74}$$

Note that even though the metric and gauge fields in (57) become singular at the background $\mu = 0 = J = Q$, we can see from the form (63) that this is just a coordinate singularity.

The parameter ζ is pure gauge because the variation $\delta\zeta$ is not present in the infinitesimal charges (73). Note however that ζ appears explicitly in the definition of the mass by associating it with the basis element $(\frac{\partial}{\partial t}, -\zeta)$. It is only in the gauge $\zeta = 0$, that the mass is associated with the time-like Killing vector $(\frac{\partial}{\partial t}, 0)$. This definition ensures in particular that the mass of the black hole does not depend on the gauge transformations (48) needed to regularize the potential on the bifurcation surfaces.

In order to compare with standard AdS black holes, one has to compute the mass in the frame (61) instead of

using the rest frame for the fluid. The conserved charge $\mathcal{E}'$ associated with $(\partial/\partial t - \alpha\partial/\partial\varphi, -\zeta + 4GQ)$ is now given by

$$\mathcal{E}' = \mathcal{E} - \alpha\mathcal{J} = \mu - \alpha J = M, \tag{75}$$

which coincides with the conventional definition of the mass for the BTZ black holes.

B. Horizon and first law

1. General derivation

When it exists, the outer horizon H is located at r_+, the largest positive root of $f(r)$. In the following, a subscript $+$ on a function means that it is evaluated at r_+. The generator of the horizon is given by $\xi = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}$, where the angular velocity Ω of the horizon has the value

$$\Omega = -\varepsilon_{h_+}\varepsilon_{q_+}\sqrt{\frac{p_+}{q_+}} = -\frac{h_+}{q_+}, \tag{76}$$

where ε_{h_+} denotes the sign of h_+. The first law can be derived by starting from

$$\begin{aligned} \delta\mathcal{E} &= \oint_S k_{\frac{\partial}{\partial t},-\zeta} \\ &= \oint_S k_{\xi,0} + \Omega\oint_S k_{-\frac{\partial}{\partial\varphi},-\frac{4GQ}{\alpha}} + \oint_S k_{-\zeta+\frac{4GQ}{\alpha}\Omega,0} \\ &= \oint_H k_{\xi,0} + \Omega\delta\mathcal{J} + (\zeta - \frac{4GQ}{\alpha}\Omega)\delta\mathcal{Q}. \end{aligned} \tag{77}$$

The first term on the right-hand side is computed using standard arguments (see for example [29, 32]) to give

$$\delta\mathcal{E} = \frac{\kappa}{8\pi G}\delta\mathcal{A} + \Omega\delta\mathcal{J} + \Phi_H^{tot}\delta Q, \tag{78}$$

where the total electric potential is given by

$$\Phi_H^{tot} = \Phi_H + \zeta - \frac{4GQ}{\alpha}\Omega,\ \Phi_H = -(\xi\cdot A)_+. \tag{79}$$

The surface gravity is given by

$$\kappa = \sqrt{|-\frac{1}{2}(D^\mu\xi^\nu)(D_\mu\xi_\nu)|}\Big|_H = \frac{|f'_+|}{2\sqrt{|q_+|}}, \tag{80}$$

and the proper area by

$$\mathcal{A} = 2\pi\sqrt{|q_+|}. \tag{81}$$

Note that the choice of signs in the definition of electric charge and angular momentum were made so that the first laws appear in the conventional form (78).

2. Explicit values and discussion

We have

$$f(r) = 2\frac{(1+\alpha^2 l^2)}{l^2}r^2 - 16G\mu\left(r - \frac{2GJ}{\alpha}\right) \tag{82}$$

so that

$$r_+ = \frac{4l^2 G\mu}{1+\alpha^2 l^2}\Big[1 + \sqrt{1 - \frac{J(1+\alpha^2 l^2)}{\alpha l^2\mu}}\Big] \tag{83}$$

In order to explicitly verify the first law (80), we start by showing that $\Phi^{tot} = 0$. We need to verify that

$$-A_t(r_+) - \Omega A_\varphi(r_+) + \zeta - \Omega\frac{4GQ}{\alpha} = 0. \tag{84}$$

Using the explicit expressions for the components of A, this equation reduces to

$$\Omega = \frac{4G\mu}{\alpha r_+}. \tag{85}$$

Taking into account $\Omega = -h_+/q_+$ together with $q_+ = h_+^2/p_+$, this equality can then easily be checked using $h_+ = -2\alpha r_+$, $p_+ = 8G\mu$, implying $q_+ = \alpha^2 r_+^2/(2G\mu)$. Since $f'_+ = 4(1+\alpha^2 l^2)r_+/l^2 - 16G\mu$, the first law reduces to

$$\delta\mu - \frac{4G\mu}{\alpha r_+}\delta J = [\frac{\alpha^2 l^2 + 1}{4Gl^2}r_+ - \mu][\frac{2\delta r_+}{r_+} - \frac{\delta\mu}{\mu}], \tag{86}$$

which can be explicitly checked using (83).

In particular, the first law (78) can be evaluated in the gauge where the potential is regular on the horizon r_+. Because the two forms (31) and (57) of the black hole solution are related by the change of coordinates $t \to t\sqrt{-8G\mu}$, $r \to r/\sqrt{-8G\mu}$, the gauge (48) now corresponds to

$$A_t = A_t^+ = -\Omega A_\varphi^+. \tag{87}$$

This amounts to the choice $\zeta = \frac{4GQ}{\alpha}\Omega$ in (57). It follows that $\Phi^{tot} = \Phi = 0$ and that the vector associated to A is proportional to ξ on the horizon.

The first law adapted to the energy $\mathcal{E}' = \mathcal{E} - \alpha\mathcal{J}$ is obtained by changing Ω to $\Omega' = \Omega - \alpha$ in (78). This form of the first law reduces to the standard form for 3 dimensional AdS black holes (with or without topological charge) when $\alpha = \pm 1/l$.

Finally, we note that the first law (78) applies both to the outer event horizon of a black hole in the normal region and to the horizon at r_0 of a cosmon, when time is identified with real period $2\pi/|\kappa|$.

Acknowledgments

M. B. wants to thank M. Henneaux for useful discussions. His work is supported by Proyectos FONDECYT

1020832 and 7020832. G. B. and G. C. are associated with the National Fund for Scientific Research, Belgium. Their work is supported in part by a "Pôle d'Attraction Interuniversitaire" (Belgium), by IISN-Belgium, convention 4.4505.86, by Proyectos FONDECYT 1970151 and 7960001 (Chile) and by the European Commission program MRTN-CT-2004-005104, in which these authors are associated to V.U. Brussels. The work of A.G. is supported in part by FONDECYT grants 1051084,1051064 and by the research grant 26-05/R of Universidad Andrés Bello.

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[33] It would be interesting to explore in details the supersymmetric properties of this action and explicitly relate it to the five dimensional supergravity action.

[34] M.B. thanks M. Henneaux for his suggestion to understand these solutions as identifications on the Gödel background.

[35] An equivalent way to do this transformation without introducing the imaginary unit is by the following sequence of coordinate transformations (and analytic continuations) acting on (12): $t \to 2t^{1/2}$, $t \to -t$, $t \to \frac{1}{4}t^2$, and the same for φ.